

# Separable $p$ -harmonic functions in a cone and related quasilinear equations on manifolds

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**Abstract** In considering a class of quasilinear elliptic equations on a Riemannian manifold with nonnegative Ricci curvature, we give a new proof of Tolksdorf's result on the construction of separable  $p$ -harmonic functions in a cone.

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*Key words.*

## 1 Introduction

Let  $(r, \sigma)$  be the spherical coordinates in  $\mathbb{R}^N$ . If  $u$  is a harmonic function in  $\mathbb{R}^N \setminus \{0\}$  written under the separable form

$$u(x) = r^{-\beta} \omega(\sigma) \quad (1.1)$$

it is straightforward to check that  $\omega$  is an eigenfunction of the Laplace-Beltrami operator  $-\Delta_{S^{N-1}}$  on the unit sphere  $S^{N-1} \subset \mathbb{R}^N$  and  $\beta$  is a root of

$$X^2 - (N-2)X - \lambda = 0, \quad (1.2)$$

where  $\lambda \geq 0$  is the corresponding eigenvalue. The function  $\omega$  is called a spherical harmonic and its properties are well-known, since such functions are the restrictions to the sphere of homogeneous harmonic polynomials. More generally, if  $C_S \subset \mathbb{R}^N$  is the cone with vertex 0 and opening  $S \subsetneq S^{N-1}$ , there exist positive harmonic functions  $u$  in  $C_S$  under the form (1.1) which vanish on  $\partial C_S \setminus \{0\}$  if and only if  $\beta$  is a root of (1.2), where, in that case,  $\lambda := \lambda_S$  is the first eigenvalue of  $-\Delta_{S^{N-1}}$  in  $W_0^{1,2}(S)$ . These separable harmonic functions play a fundamental role in the description of isolated interior or boundary singularities of solutions of second order linear elliptic equations. If the Laplace equation is replaced by the  $p$ -Laplace equation

$$-\Delta_p u := -\operatorname{div}(|Du|^{p-2} Du) = 0, \quad (1.3)$$

( $p > 1$ ), the same question of existence of separable  $p$ -harmonic functions, i.e. solutions of (1.3) in the form (1.1), was considered by Krol [11], Tolksdorf [19], Kichenassamy and Véron [10]. If  $u$  in (1.1) is  $p$ -harmonic, then the function  $\omega$  must be a solution of the spherical  $p$ -harmonic equation,

$$- \operatorname{div} \left( (\beta^2 \omega^2 + |\nabla' \omega|^2)^{p/2-1} \nabla' \omega \right) = \beta(\beta(p-1) + p - N) (\beta^2 \omega^2 + |\nabla' \omega|^2)^{p/2-1} \omega, \quad (1.4)$$

on  $S^{N-1}$ , where  $\nabla'$  and  $\operatorname{div}$  are respectively the covariant derivative identified with the “tangential gradient” and the divergence operator acting on vector fields on  $S^{N-1}$ . Two special cases arise when either  $p = 2$  or  $N = 2$ : if  $p = 2$ , (1.4) is just an eigenvalue problem

$$- \Delta' \omega = \beta(\beta + 2 - N) \omega, \quad (1.5)$$

where  $\Delta'$  is the Laplace-Beltrami operator on  $S^{N-1}$ . When  $N = 2$ , equation (1.4) becomes

$$- \left( (\beta^2 \omega^2 + |\omega_\theta|^2)^{p/2-1} \omega_\theta \right)_\theta = \beta(\beta(p-1) + p - 2) (\beta^2 \omega^2 + |\omega_\theta|^2)^{p/2-1} \omega, \quad (1.6)$$

where  $\theta \in [0, \pi]$ . Introducing the new unknown  $\phi := \omega_\theta / \omega$ , (1.6) is transformed into a separable equation,

$$- \left( (\beta^2 + \phi^2)^{p/2-1} \phi \right)_\theta = ((p-1)\phi^2 + \beta(\beta(p-1) + p - 2)) (\beta^2 + \phi^2)^{p/2-1}. \quad (1.7)$$

This equation was completely integrated by Krol [11] in the case  $\beta < 0$ , and Kichenassamy and Véron [10] in the case  $\beta > 0$ . It turns out that for any integer  $k > 0$  there exist two couples  $(\tilde{\beta}_k, \tilde{\phi}_k)$  and  $(\beta_k, \phi_k)$  where  $\tilde{\beta}_k < 0$ ,  $\beta_k > 0$ , and  $\tilde{\phi}_k$  and  $\phi_k$  are anti-periodic solutions of the corresponding equation (1.7). Furthermore  $\tilde{\phi}_k$  and  $\phi_k$  are uniquely determined, up to an homothety.

A remarkable breakthrough was realized by Tolksdorf [19] when he proved that for any smooth domain  $S \subset S^{N-1}$  there exists a couple  $(\beta, \phi)$  where  $\beta < 0$  and  $\phi \in C^1(\bar{S})$  is positive in  $S$ , vanishes on  $\partial S$  and solves (1.4) in  $S$ . Furthermore  $\beta := \tilde{\beta}_S$  is unique and  $\phi$  is determined up to a multiplicative constant. Tolksdorf’s result is obtained by constructing a  $p$ -harmonic function  $u$  in the cone  $C_S$  generated by  $S$  with a compactly supported boundary data and by proving, thanks to a kind of Harnack inequality up to the boundary, the “equivalence principle”, that the asymptotic behaviour of  $u$  is self-similar. Later on the existence of a couple  $(\beta, \phi)$ , with  $\beta := \beta_S > 0$  and  $\phi$ , as above, positive solution of (1.4) in  $S$  vanishing on  $\partial S$  is proved by the same method in [21], therefore we shall refer to the two cases  $\beta > 0$  and  $\beta < 0$  as Tolksdorf’s results. The structure of these spherical  $p$ -harmonic functions is studied in [5] when  $p = N$ . These regular ( $\beta < 0$ ) and singular ( $\beta > 0$ ) separable  $p$ -harmonic functions play a fundamental role in describing the behaviour of solutions of quasilinear equations near a regular or singular boundary point [11],[12],[3],[6].

In this article, we give a new proof of Tolksdorf’s results, entirely different from his. Actually, performing a change of variable, we embed our problem into a much wider class

of quasilinear equations. Indeed, if  $\omega \in W_0^{1,p}(S)$  is a positive solution of (1.4) in  $S \subset S^{N-1}$ , which vanishes on  $\partial S$ , then the function  $v$  defined by

$$v = -\frac{1}{\beta} \ln \omega$$

solves

$$\begin{cases} -\operatorname{div} \left( (1 + |\nabla' v|^2)^{p/2-1} \nabla' v \right) + \beta(p-1) (1 + |\nabla' v|^2)^{p/2-1} |\nabla' v|^2 \\ \qquad \qquad \qquad = -(\beta(p-1) + p - N) (1 + |\nabla' v|^2)^{p/2-1} & \text{in } S \\ \lim_{\sigma \rightarrow \partial S} v(\sigma) = \infty. \end{cases} \quad (1.8)$$

Notice that this equation is never degenerate and  $v$  is  $C^2$  (actually  $C^\infty$ ) in  $S$  and satisfies the equation and the boundary condition in classical sense. Our construction of solutions of (1.4) relies on a careful study of the quasilinear problem (1.8), and on the interpretation of the constant in the right hand side of (1.8) as an “ergodic constant”. Furthermore, having an intrinsic independent interest, this study will be performed on any compact smooth subdomain of a Riemannian manifold, without referring to the p-Laplace equation (1.3). Our main result is the following:

**Theorem A.** *Let  $(M, g)$  be a  $d$ -dimensional Riemannian manifold with nonnegative Ricci curvature, and let  $\nabla$  and  $\operatorname{div}_g$  be respectively the covariant derivative and the divergence operator on  $M$ . Then for any compact smooth subdomain  $S \subset M$  and any  $\beta > 0$  there exists a unique positive constant  $\lambda_\beta$  such that the problem*

$$\begin{cases} -\operatorname{div}_g \left( (1 + |\nabla v|^2)^{p/2-1} \nabla v \right) + \beta(p-1) (1 + |\nabla v|^2)^{p/2-1} |\nabla v|^2 \\ \qquad \qquad \qquad = -\lambda_\beta (1 + |\nabla v|^2)^{p/2-1} & \text{in } S \\ \lim_{x \rightarrow \partial S} v(x) = \infty. \end{cases} \quad (1.9)$$

*admits a solution  $v \in C^2(S)$ . Furthermore,  $v$  is unique up to an additive constant.*

The result of Theorem A is the typical statement of an ergodic problem, indeed the constant  $\lambda_\beta$  can be seen as the unique ergodic constant for the equation obtained after dividing by  $(1 + |\nabla v|^2)^{p/2-1}$  (see (2.1)). Observe also that (1.9) may be reformulated if we set  $\omega = e^{-\beta v}$ , then  $\omega$  is a solution of

$$\begin{cases} -\operatorname{div}_g \left( (\beta^2 \omega^2 + |\nabla \omega|^2)^{p/2-1} \nabla \omega \right) = \beta \lambda_\beta (\beta^2 \omega^2 + |\nabla \omega|^2)^{p/2-1} \omega & \text{in } S \\ \omega = 0 & \text{on } \partial S \end{cases} \quad (1.10)$$

When  $p = 2$ , problem (1.10) reduces to an eigenvalue problem since  $\beta \lambda_\beta = \lambda_1(S)$ , the principal eigenvalue of the Laplace–Beltrami operator in  $S$ . In that case the connection between (1.9) and (1.10) dates back to the stochastic interpretation of principal eigenvalues (see e.g. [16], [17]). In the nonlinear framework with  $p \neq 2$ , by proving that the

mapping  $\beta \mapsto \lambda_\beta$  is continuous, decreasing and tends to  $\infty$  as  $\beta \rightarrow 0^+$ , we conclude that the equation  $\lambda_\beta = (\beta(p-1) + p - d - 1)$  has a unique positive solution. As a consequence we generalize Tolksdorf's result as follows.

**Theorem B.** *Under the assumptions of Theorem A, for any compact smooth subdomain  $S$  of  $M$  there exists a unique  $\beta := \beta_S > 0$  such that the problem*

$$\begin{cases} -\operatorname{div}_g ((\beta^2 \omega^2 + |\nabla \omega|^2)^{p/2-1} \nabla \omega) = \beta (\beta(p-1) + p - d - 1) (\beta^2 \omega^2 + |\nabla \omega|^2)^{p/2-1} \omega & \text{in } S \\ \omega = 0 & \text{on } \partial S, \end{cases} \quad (1.11)$$

*admits a positive solution  $\omega \in C^1(\bar{S}) \cap C^2(S)$ . Furthermore  $\omega$  is unique up to an homothethy.*

Of course, we obtain similarly that for  $\beta < 0$  there exists a unique  $\beta := \tilde{\beta}_S < 0$  such that  $\lambda_\beta = (\beta(p-1) + p - d - 1)$ . Tolksdorf's results then follow as a particular case by taking  $(M, g) = (S^{N-1}, g_0)$ , where  $S^{N-1}$  is equipped with the standard metric  $g_0$  induced by the Euclidean structure in  $\mathbb{R}^N$ .

## 2 The singular case

In the following, we consider a general geometric setting and we recall some elements of Riemannian geometry (see e.g. [13], [15]). Let  $(M, g)$  be a complete  $d$ -dimensional Riemannian manifold with metric tensor  $g = (g_{ij})$ , inverse  $g^{-1} = (g^{ij})$  and determinant  $|g|$ . If  $X$  and  $Y$  are two tangent vector fields to  $M$ , we denote by

$$X.Y = \sum_{ij} g_{ij}(x) X^i Y^j$$

their scalar product in the tangent space  $T_x M$ . Let  $x_j$ ,  $j = 1, \dots, d$ , be a local system of coordinates: if  $u \in C^1(M)$ , the gradient of  $u$ , quoted by  $\nabla u$ , is the vector field with components  $(\nabla u)^i = \sum_k g^{ik} u_{x_k}$ . Therefore

$$\nabla u . \nabla u = |\nabla u|^2 = \sum_{ij} g^{ij}(x) u_{x_i} u_{x_j}.$$

If  $X = (X^i)$  is a  $C^1$  vector field on  $M$ , the divergence of  $X$  is defined by

$$\operatorname{div}_g X = \frac{1}{\sqrt{|g|}} \sum_k \left( \sqrt{|g|} X^k \right)_{x_k}.$$

Recalling that, in local coordinates, the Christoffel symbols are

$$\Gamma_{ij}^k = \frac{1}{2} \sum_l \left( \frac{\partial g_{jl}}{\partial x_i} + \frac{\partial g_{li}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_l} \right) g^{lk},$$

the second covariant derivatives of a  $C^2$  function  $u$  are

$$\nabla_{ij} u = u_{x_i x_j} - \sum_k \Gamma_{ij}^k u_{x_k},$$

while the Hessian is the 2-tensor  $D^2u = (\nabla_{ij}u)$ . Finally,  $\Delta_g u = \text{trace}(D^2u) = \text{div}_g \nabla u$  is the Laplace-Beltrami operator on  $M$ , locally expressed by

$$\Delta_g u = \frac{1}{\sqrt{|g|}} \sum_{ij} \frac{\partial}{\partial x_i} \left( \sqrt{|g|} g^{ij} \frac{\partial u}{\partial x_j} \right) = \sum_{ij} \frac{\partial}{\partial x_i} \left( g^{ij} \frac{\partial u}{\partial x_j} \right) + \sum_{ijk} \Gamma_{ik}^k g^{ij} \frac{\partial u}{\partial x_j}.$$

We denote by  $\text{Ric}_g$  the Ricci curvature tensor of the metric  $g$ . In particular, if  $(M, g) = (S^{N-1}, g_0)$ , then  $\text{Ric}_{g_0} = (N-1)g_0$ .

In all the sequel  $p > 1$  is a real number. We prove next the result of Theorem A, which we restate here for the reader's convenience.

**Theorem 2.1** *Let  $S \subset M$  be a smooth bounded open domain of  $M$  such that  $\text{Ric}_g \geq 0$  on  $S$ . Then for any  $\beta > 0$  there exists a unique  $\lambda_\beta > 0$  such that there exists a function  $v \in C^2(S)$  satisfying*

$$\begin{cases} -\Delta_g v - (p-2) \frac{D^2 v \nabla v \cdot \nabla v}{1 + |\nabla v|^2} + \beta(p-1) |\nabla v|^2 = -\lambda_\beta & \text{in } S \\ \lim_{x \rightarrow \partial S} v(x) = \infty. \end{cases} \quad (2.1)$$

Furthermore,  $v$  is unique up to an additive constant.

*Proof.* As in the usual approach to ergodic problems, we start by considering the problem

$$\begin{cases} -\Delta_g v_\epsilon - (p-2) \frac{D^2 v_\epsilon \nabla v_\epsilon \cdot \nabla v_\epsilon}{1 + |\nabla v_\epsilon|^2} + \beta(p-1) |\nabla v_\epsilon|^2 + \epsilon v_\epsilon = 0 & \text{in } S \\ \lim_{x \rightarrow \partial S} v_\epsilon(x) = \infty, \end{cases} \quad (2.2)$$

where  $\epsilon > 0$ , and then we study the limit when  $\epsilon \rightarrow 0$ .

*Step 1: Construction of super and sub solutions.* Since  $\partial S$  is  $C^2$ , the distance function  $\rho(x) = \text{dist}(x, \partial S)$ , where the distance is the geodesic distance, is a positive  $C^2$  function in some relative neighborhood  $\mathcal{N}_\delta = \{x \in M : |\rho(x)| < \delta\}$  of  $\partial S$ ; here  $\dot{\rho}(x)$  is the signed distance, equal to  $\pm \rho(x)$  according  $x \in S$  or  $x \in M \setminus S$ . Then  $|\nabla \dot{\rho}(x)| = 1$  in  $\mathcal{N}_\delta$ . We extend  $\dot{\rho}$  outside  $\mathcal{N}_\delta$  into a  $C^2(M)$  function  $\tilde{\rho}$ . Next we consider the function

$$\bar{u}(x) = -\frac{1}{\beta} \ln(\tilde{\rho}(x)) - M_0 \tilde{\rho}(x) + \frac{M_1}{\epsilon} \quad \forall x \in S, \quad (2.3)$$

where the  $M_j > 0$  are to be chosen later on. Then

$$\begin{aligned} \nabla \bar{u}(x) &= -\frac{1}{\beta \tilde{\rho}(x)} (1 + \beta M_0 \tilde{\rho}(x)) \nabla \tilde{\rho}(x), \\ |\nabla \bar{u}(x)|^2 &= \frac{1}{\beta^2 \tilde{\rho}^2(x)} (1 + \beta M_0 \tilde{\rho}(x))^2 |\nabla \tilde{\rho}(x)|^2. \end{aligned}$$

Notice that this last identity implies

$$|\nabla \bar{u}(x)|^2 = \frac{1 + 2\beta M_0 \rho(x) + O(\rho^2(x))}{\beta^2 \rho^2(x)} \quad \text{as } \rho(x) \rightarrow 0.$$

Next

$$\begin{aligned}
-\Delta_g \bar{u} - (p-2) \frac{D^2 \bar{u} \nabla \bar{u} \cdot \nabla \bar{u}}{1 + |\nabla \bar{u}|^2} &= -\Delta_g \bar{u} - \frac{(p-2)}{2} \frac{\nabla(|\nabla \bar{u}|^2) \cdot \nabla \bar{u}}{1 + |\nabla \bar{u}|^2} \\
&= -\frac{|\nabla \tilde{\rho}|^2}{\beta \tilde{\rho}^2} + \frac{\Delta_g \tilde{\rho}}{\beta \tilde{\rho}} (1 + \beta M_0 \tilde{\rho}) - \frac{(p-2)}{\beta \tilde{\rho}^2} \frac{|\nabla \tilde{\rho}|^4 (1 + \beta M_0 \tilde{\rho})^3}{\beta^2 \tilde{\rho}^2 + |\nabla \tilde{\rho}|^2 (1 + \beta M_0 \tilde{\rho})^2} \\
&\quad + \frac{(p-2)}{2\beta \tilde{\rho}} \frac{(1 + \beta M_0 \tilde{\rho}) \nabla \tilde{\rho} \cdot \nabla [|\nabla \tilde{\rho}|^2 (1 + \beta M_0 \tilde{\rho})^2]}{\beta^2 \tilde{\rho}^2 + |\nabla \tilde{\rho}|^2 (1 + \beta M_0 \tilde{\rho})^2}.
\end{aligned}$$

After some lengthy but standard computations, one obtains the following relation

$$\begin{aligned}
-\Delta_g \bar{u} - (p-2) \frac{D^2 \bar{u} \nabla \bar{u} \cdot \nabla \bar{u}}{1 + |\nabla \bar{u}|^2} + \beta(p-1)|\nabla \bar{u}|^2 + \epsilon \bar{u} \\
= \frac{1}{\tilde{\rho}} \left( \frac{\Delta_g \tilde{\rho}}{\beta} - \frac{\epsilon}{\beta} \tilde{\rho} \ln(\tilde{\rho}) + 2(p-1)M_0 |\nabla \tilde{\rho}|^2 \right) + \psi_\beta(x) + M_1,
\end{aligned} \tag{2.4}$$

where  $\psi_\beta$  is a function depending on  $\beta$  (and on  $M_0$ ), but which remains bounded on  $S$ , uniformly when  $\beta$  remains in a compact subset of  $(0, \infty)$ . Since  $|\nabla \tilde{\rho}| = 1$  near the boundary, it is possible to choose  $M_0$  and  $M_1$  such that  $\bar{u}$  defined by (2.3) is a supersolution for (2.2). Moreover,  $M_0$  and  $M_1$  can be chosen independent of  $\beta$  whenever it varies on a compact subset of  $(0, \infty)$ .

One finds similarly that the function

$$\underline{u}(x) = -\frac{1}{\beta} \ln(\tilde{\rho}(x)) + M_0 \tilde{\rho}(x) - \frac{M_1}{\epsilon} \quad \forall x \in S, \tag{2.5}$$

is a subsolution of (2.2), with  $M_0$  and  $M_1$  chosen as for  $\bar{u}$ . Moreover, for  $0 < h < \delta$ , we can approximate  $\bar{u}$  and  $\underline{u}$  respectively from above and from below by

$$\bar{u}_h(x) = -\frac{1}{\beta} \ln(\tilde{\rho}(x) - h) - M_0(\tilde{\rho}(x) - h) + \frac{M_{1,h}}{\epsilon}, \tag{2.6}$$

$$\underline{u}_h(x) = -\frac{1}{\beta} \ln(\tilde{\rho}(x) + h) + M_0(\tilde{\rho}(x) + h) - \frac{M_{1,h}}{\epsilon}, \tag{2.7}$$

which are, respectively, a supersolution in  $\{x \in S : \rho(x) > h\}$  and a subsolution in  $S$ . Together with the comparison principle, these super and sub solutions will be used to derive estimates on the solutions of (2.2).

*Step 2: Basic estimates.* In this part, by using the classical Bernstein's method ([2]), we derive the fundamental gradient estimate for the solutions  $u \in C^2(S)$  of

$$-\Delta_g u - (p-2) \frac{D^2 u \nabla u \cdot \nabla u}{1 + |\nabla u|^2} + \beta(p-1)|\nabla u|^2 + \epsilon u = 0 \quad \text{in } S. \tag{2.8}$$

We recall the Weitzenböck formula (see e.g. [1]):

$$\frac{1}{2} \Delta_g |\nabla u|^2 = |D^2 u|^2 + \nabla(\Delta_g u) \cdot \nabla u + Ricc_g(\nabla u, \nabla u), \tag{2.9}$$

and the Cauchy-Schwarz inequality for  $D^2u$

$$|D^2u|^2 \geq \frac{1}{d} |\Delta_g u|^2.$$

Let  $m = \inf\{Ric_g(\nabla u, \nabla u) : |\nabla u| = 1\} \geq 0$ , then

$$\frac{1}{2} \Delta_g |\nabla u|^2 \geq \frac{1}{d} |\Delta_g u|^2 + m |\nabla u|^2 + \nabla(\Delta_g u) \cdot \nabla u. \quad (2.10)$$

If we set  $z = |\nabla u|^2$ , we can re-write (2.8) as

$$\Delta_g u = -\frac{(p-2)}{2} \frac{\nabla z \cdot \nabla u}{1 + |\nabla u|^2} + \beta(p-1)z + \epsilon u \quad \text{in } S. \quad (2.11)$$

Using the fact that

$$\nabla(\nabla z \cdot \nabla u) \cdot \nabla u = D^2 z \nabla u \cdot \nabla u + \frac{1}{2} |\nabla z|^2,$$

we obtain

$$\begin{aligned} \nabla(\Delta_g u) \cdot \nabla u = & -\frac{(p-2)}{2} \frac{D^2 z \nabla u \cdot \nabla u}{1 + |\nabla u|^2} - \frac{(p-2)}{4} \frac{|\nabla z|^2}{1 + |\nabla u|^2} + \frac{(p-2)}{2} \frac{(\nabla z \cdot \nabla u)^2}{(1 + |\nabla u|^2)^2} \\ & + \beta(p-1) \nabla z \cdot \nabla u + \epsilon z. \end{aligned}$$

Since, from (2.11)

$$|\Delta_g u|^2 \geq c_0 z^2 - c_1 \left( (\epsilon u^-)^2 + \frac{(\nabla z \cdot \nabla u)^2}{(1 + |\nabla u|^2)^2} \right),$$

we derive from (2.10)

$$\begin{aligned} \Delta_g z + (p-2) \frac{D^2 z \nabla u \cdot \nabla u}{1 + |\nabla u|^2} \geq & \frac{2c_0 z^2}{d} - \frac{2c_1}{d} \left( (\epsilon u^-)^2 + \frac{(\nabla z \cdot \nabla u)^2}{(1 + |\nabla u|^2)^2} \right) + 2(m + \epsilon)z \\ & - \frac{(p-2)}{2} \frac{|\nabla z|^2}{1 + |\nabla u|^2} + (p-2) \frac{(\nabla z \cdot \nabla u)^2}{(1 + |\nabla u|^2)^2} + 2\beta(p-1) \nabla z \cdot \nabla u, \end{aligned}$$

which yields, by Young's inequality and the fact that  $z = |\nabla u|^2$ ,

$$-\Delta_g z - (p-2) \frac{D^2 z \nabla u \cdot \nabla u}{1 + |\nabla u|^2} + C_0 z^2 + 2(m + \epsilon)z \leq C_1 \frac{|\nabla z|^2}{1 + z} + C_2 \quad (2.12)$$

for some positive constants  $C_j$  ( $j = 0, 1, 2$ ), eventually depending on  $\beta$ , with the constant  $C_2$  also depending on  $\|\epsilon u^-\|_\infty$ . Next we introduce the operator  $\mathcal{A}$  defined by

$$\mathcal{A}(z) = -\Delta_g z - (p-2) \frac{D^2 z \nabla u \cdot \nabla u}{1 + |\nabla u|^2}, \quad (2.13)$$

which can be written, in local coordinates, as

$$\begin{aligned}
\mathcal{A}(z) &= - \sum_{ij} g^{ij} z_{x_i x_j} - (p-2) \sum_{ij} \frac{\sum_{kl} g^{ik} u_{x_k} g^{jl} u_{x_l}}{1 + \sum_{kl} g^{kl} u_{x_k} u_{x_l}} z_{x_i x_j} \\
&\quad - \sum_{ijk} \left( \left( \Gamma_{ik}^k g^{ij} + g_{x_i}^{ij} \right) z_{x_j} + (p-2) \Gamma_{ij}^k \frac{\sum_{lm} g^{im} u_{x_m} g^{jl} u_{x_l}}{1 + \sum_{lm} g^{ml} u_{x_m} u_{x_l}} z_{x_k} \right) \\
&= - \sum_{ij} a_{ij} z_{x_i x_j} + \sum_i b_i z_{x_i},
\end{aligned} \tag{2.14}$$

where the  $a_{ij}$  are uniformly elliptic and bounded and the  $b_i$  are bounded: indeed, it holds

$$\min(p-1, 1) g^{ij} \xi_i \xi_j \leq a_{ij} \xi_i \xi_j \leq \max(1, p-1) g^{ij} \xi_i \xi_j.$$

Therefore from (2.12)  $z$  is a positive subsolution of an equation of the type

$$\mathcal{A}(z) + h(z) + g(z) |\nabla z|^2 = f, \tag{2.15}$$

where  $g(z) = -C_1(1+z)^{-1}$ ,  $h(z) = 2(m+\epsilon)z + C_0 z^2$  and  $f = C_2$ . Since  $m \geq 0$ ,  $g$  and  $h$  are increasing functions of the nonnegative variable  $z$ , it follows that the comparison principle holds between super and sub-solutions of

$$- \Delta_g z - (p-2) \frac{D^2 z \nabla u \cdot \nabla u}{1 + |\nabla u|^2} + C_0 z^2 + 2(m+\epsilon)z - C_1 \frac{|\nabla z|^2}{1+z} = C_2. \tag{2.16}$$

Standard computations show that, if  $\lambda$  and  $\mu$  are positive constants large enough, the function

$$\bar{z}(x) = \frac{\lambda}{\tilde{\rho}^2(x)} + \mu$$

is a supersolution of (2.16), which in addition blows up on  $\partial S$ . We conclude that any bounded subsolution of (2.16) satisfies  $z(x) \leq \bar{z}(x)$ , and therefore any subsolution by replacing  $S$  by  $\{x \in S : \rho(x) > h\}$  and  $\tilde{\rho}(x)$  by  $\tilde{\rho}(x) - h$ .

Finally, we proved that any  $u \in C^2(S)$  which is solution of (2.8) satisfies

$$|\nabla u(x)| \leq \frac{L_0}{\tilde{\rho}(x)} + L_1 \quad \forall x \in S, \tag{2.17}$$

for some constants  $L_0, L_1$  depending on  $\|\varepsilon u^-\|_\infty$ . Moreover,  $L_0$  and  $L_1$  can be chosen uniformly bounded with respect to  $\beta$ , provided  $\beta$  remains in a compact subset of  $(0, \infty)$ .

To conclude with the estimates on solutions of (2.8), it is classical from the theory of quasilinear elliptic equations (see e.g. [14]) that local Lipschitz estimates imply local  $C^{2,\alpha}$  estimates since the equation is smooth and uniformly elliptic.



*Step 3: Existence for the approximate equation.* As in [17], we consider, for  $n \in \mathbb{N}$  the solution  $v_{n,\epsilon} := v$  of

$$\begin{cases} -\Delta_g v - (p-2) \frac{D^2 v \nabla v \cdot \nabla v}{1 + |\nabla v|^2} + \beta(p-1) |\nabla v|^2 + \epsilon v = 0 & \text{in } S \\ v(x) = n & \text{on } \partial S, \end{cases} \quad (2.18)$$

By previous steps, the following estimates hold in  $S$ .

$$0 \leq v_{n,\epsilon}(x) \leq -\frac{1}{\beta} \ln \tilde{\rho}(x) - M_0 \tilde{\rho}(x) + \frac{M_1}{\epsilon}, \quad (2.19)$$

$$|\nabla v_{n,\epsilon}(x)| \leq \frac{L_0}{\tilde{\rho}(x)} + L_1. \quad (2.20)$$

Moreover the sequence  $\{v_{n,\epsilon}\}$  is bounded in  $C_{loc}^{2,\alpha}(S)$ , which ensures the local compactness of the gradients. Since  $n \mapsto v_{n,\epsilon}$  is increasing, there exists  $v_\epsilon = \lim_{n \rightarrow \infty} v_{n,\epsilon}$  and  $v_\epsilon$  is a solution of (2.2) which satisfies (2.19) and (2.20).

*Step 4: The ergodic limit.* From Step 1, by comparison with  $\bar{u}_h$  and  $\underline{u}_h$  defined in (2.6)–(2.7) (and letting  $h \rightarrow 0$ ), we know that their holds in  $S$ :

$$-\frac{1}{\beta} \ln \tilde{\rho}(x) + M_0 \tilde{\rho}(x) - \frac{M_1}{\epsilon} \leq v_\epsilon(x) \leq -\frac{1}{\beta} \ln \tilde{\rho}(x) - M_0 \tilde{\rho}(x) + \frac{M_1}{\epsilon}. \quad (2.21)$$

Therefore  $\epsilon v_\epsilon$  is locally bounded in  $S$ . Since  $\nabla v_\epsilon$  is locally bounded too in  $S$ ,  $\epsilon_n v_{\epsilon_n}$  converges to some constant  $\lambda_0 \geq 0$  for some sequence  $\{\epsilon_n\}$  in the  $C_{loc}$ -topology of  $S$ . We fix  $x_0 \in S$  and set  $w_\epsilon := v_\epsilon(x) - v_\epsilon(x_0)$ . Because  $w_\epsilon$  is locally bounded in  $C_{loc}^1(S)$  and  $w_\epsilon$  satisfies

$$-\Delta_g w_\epsilon - (p-2) \frac{D^2 w_\epsilon \nabla w_\epsilon \cdot \nabla w_\epsilon}{1 + |\nabla w_\epsilon|^2} + \beta(p-1) |\nabla w_\epsilon|^2 + \epsilon w_\epsilon = -\epsilon v_\epsilon(x_0) \quad \text{in } S \quad (2.22)$$

the regularity theory for elliptic equations implies that  $w_\epsilon$  is locally bounded in  $C^{2,\alpha}(S)$ . Up to an extraction of subsequence, there exists  $w_0 = \lim_{n \rightarrow \infty} w_{\epsilon_n}$ , and  $w_0$  is a solution of

$$-\Delta_g w_0 - (p-2) \frac{D^2 w_0 \nabla w_0 \cdot \nabla w_0}{1 + |\nabla w_0|^2} + \beta(p-1) |\nabla w_0|^2 = -\lambda_0 \quad \text{in } S. \quad (2.23)$$

The only question which remains to be proved is that  $w_0$  blows-up at the boundary. We set

$$\underline{\psi}(x) = -\frac{1}{\beta} \ln \tilde{\rho}(x) + M_0 \tilde{\rho}(x),$$

and get, with same computations as in (2.4),

$$\begin{aligned} -\Delta_g \underline{\psi} - (p-2) \frac{D^2 \underline{\psi} \nabla \underline{\psi} \cdot \nabla \underline{\psi}}{1 + |\nabla \underline{\psi}|^2} + \beta(p-1) |\nabla \underline{\psi}|^2 + \epsilon \underline{\psi} \\ = \frac{1}{\tilde{\rho}} \left( \frac{\Delta_g \tilde{\rho}}{\beta} - \frac{\epsilon}{\beta} \tilde{\rho} \ln(\tilde{\rho}) - 2(p-1) M_0 |\nabla \tilde{\rho}|^2 \right) + \psi_\beta(x), \end{aligned} \quad (2.24)$$

where  $\psi_\beta$  is a bounded function (depending on  $\beta$ ,  $M_0$ ). Noticing that  $|\nabla \tilde{\rho}| = 1$  in a neighborhood of  $\partial S$ , and that  $\epsilon v_\epsilon(x_0)$  is uniformly bounded, we can choose  $M_0, \rho_0$  such that the function  $\underline{\psi}$  is a subsolution of (2.22) in  $\{x \in S : 0 < \rho(x) < \rho_0\}$ . Since, whenever  $\rho(x) = \rho_0$ , we have  $w_\epsilon(x) \geq -c_0$  for some  $c_0 > 0$  (due to the gradient estimate for  $v_\epsilon$ ), and since  $\underline{\psi} - c$  is still a subsolution for any positive constant  $c$ , we derive

$$w_\epsilon(x) \geq -\frac{1}{\beta} \ln \tilde{\rho}(x) + M_0 \tilde{\rho}(x) - c \quad \forall x \text{ s.t. } \rho(x) \leq \rho_0. \quad (2.25)$$

Letting  $\epsilon$  tend to 0 implies that  $\lim_{x \rightarrow \partial S} w_0(x) = \infty$ .

*Step 5: Uniqueness of the ergodic limit.* We claim that there exists a unique constant  $\lambda_0 > 0$  such that there exists  $v_0 \in C^2(S)$  solution of

$$\begin{cases} -\Delta_g v_0 - (p-2) \frac{D^2 v_0 \nabla v_0 \cdot \nabla v_0}{1 + |\nabla v_0|^2} + \beta(p-1) |\nabla v_0|^2 = -\lambda_0 & \text{in } S \\ \lim_{x \rightarrow \partial S} v_0(x) = \infty. \end{cases} \quad (2.26)$$

To this purpose, it will be useful the following

**Lemma 2.2** *A function  $v_0 \in C^2(S)$  is solution of (2.26) if and only if the function  $\omega_0 = e^{-\beta v_0} \in C^2(S) \cap C(\bar{S})$  is a solution of*

$$\begin{cases} -\operatorname{div}_g ((\beta^2 \omega_0^2 + |\nabla \omega_0|^2)^{p/2-1} \nabla \omega_0) = \beta \lambda_0 (\beta^2 \omega_0^2 + |\nabla \omega_0|^2)^{p/2-1} \omega_0 & \text{in } S \\ \omega_0 = 0 & \text{on } \partial S. \end{cases} \quad (2.27)$$

Moreover,  $\omega_0 \in C^{1,\gamma}(\bar{S})$  for some  $\gamma > 0$ , and  $\partial_\nu \omega_0 < 0$  on  $\partial S$ .

**Proof.** Let  $v_0 \in C^2(S)$  be a solution of (2.26). As in the previous steps, considering the functions

$$\underline{\phi}(x) = -\frac{1}{\beta} \ln \tilde{\rho}(x) + M_0 \tilde{\rho}(x) - M^* \quad \text{and} \quad \bar{\phi}(x) = -\frac{1}{\beta} \ln \tilde{\rho}(x) - M_0 \tilde{\rho}(x) + M^*,$$

which appear to be respectively a sub and a super-solution for (2.26) in  $\{x : \rho(x) < \delta\}$  for some  $\delta > 0$  small enough (where  $M^*$  depends on the value of  $v_0$  on the set  $\{x \in S : \rho(x) = \delta\}$ ), we obtain

$$\left| v_0(x) + \frac{\ln \tilde{\rho}(x)}{\beta} \right| \leq M^*. \quad (2.28)$$

By the gradient estimates of Step 2, there holds

$$|\nabla v_0(x)| \leq \frac{L_0}{\tilde{\rho}(x)} + L_1. \quad (2.29)$$

Now set  $\omega_0 = e^{-\beta v_0}$ , then  $\omega_0 \in W^{1,\infty}(S) \cap C(\bar{S})$  solves the problem (2.27). By the regularity theory for degenerate equations of  $p$ -Laplacian type (see the Appendix, Theorem A.1 and related references), we can deduce that  $\omega_0 \in C^{1,\gamma}(\bar{S})$ . Moreover, since (2.28) implies

$$e^{-\beta M^*} \leq \frac{\omega_0}{\rho(x)} \leq e^{\beta M^*} \quad (2.30)$$

we deduce that  $\partial_\nu \omega_0 < -e^{-\beta M^*} < 0$  on  $\partial S$ . As a consequence, since  $\omega_0 \in C^1(\bar{S})$  and is positive in  $S$ , we deduce that problem (2.27) is uniformly elliptic, so that the classical regularity theory applies to give  $\omega_0 \in C^{2,\alpha}(S)$ .

Of course, the converse is also true: given a solution  $\omega_0$  of (2.27), clearly  $v_0 = -\frac{1}{\beta} \ln \omega_0$  is a solution of (2.26).  $\blacksquare$

Assume now that there exist two ergodic constants,  $\lambda_1$  and  $\lambda_2$ , associated with two solutions  $v_1, v_2$ , and let correspondingly  $\omega_i = e^{-\beta v_i}$  be solutions of (2.27). Notice that multiplying (2.27) by  $\omega_0$  and integrating on  $S$ , we get actually  $\lambda_0 > 0$ . Thus  $\lambda_i > 0$  and, say,  $\lambda_2 > \lambda_1$ .

Since  $\omega_1/\omega_2 \in L^\infty(S)$  (from estimate (2.30)), we denote

$$\theta = \sup_S \frac{\omega_1}{\omega_2}.$$

Because equation (2.27) is homogeneous we can assume that  $\theta = 1$  and either there exists  $x_0 \in S$  such that  $\omega_1(x_0) = \omega_2(x_0)$ ,  $\nabla \omega_1(x_0) = \nabla \omega_2(x_0)$  and  $\omega_1(x) \leq \omega_2(x)$  for  $x \in \bar{S}$ , or  $\omega_1(x) < \omega_2(x)$  for  $x \in S$  and there exists  $x_0 \in \partial S$  such that  $\partial_\nu \omega_1(x_0) = \partial_\nu \omega_2(x_0)$ . In the first case, it turns out that the function  $z = v_1 - v_2$  is nonnegative in  $S$ , achieves a minimum at  $x_0 \in S$  and satisfies

$$-\Delta_g z(x_0) - (p-2) \frac{D^2 z(x_0) \nabla v_1(x_0) \cdot \nabla v_1(x_0)}{1 + |\nabla v_1(x_0)|^2} = \lambda_2 - \lambda_1 > 0,$$

which is impossible because of ellipticity. In the second case, we have  $\partial_\nu(\omega_1 - \omega_2)(x_0) = 0$ , whereas  $\omega_1 - \omega_2$  is negative in  $S$  and  $(\omega_1 - \omega_2)(x_0) = 0$ . Since the problem (2.27) is uniformly elliptic (recall that the functions  $\omega_i$  satisfy  $(\beta^2 \omega_i^2 + |\nabla \omega_i|^2) > 0$  on  $\bar{S}$ ) this contradicts Hopf maximum principle. Therefore  $\omega_1 = \omega_2$ , which implies  $\lambda_1 = \lambda_2$  by the equation. Thus the ergodic constant is unique.

In a similar way one can prove that  $\omega_0$  is unique up to a multiplicative constant, and so  $v_0$  is unique up to an additive constant (as a consequence, the whole sequence  $w_\epsilon$ , constructed in Step 4, converges to  $w_0$  as  $\epsilon \rightarrow 0$ ).

However, the uniqueness of  $v_0$  can be proved with a more general argument, concerning directly problem (2.26), which is a variant as well as a generalization of previous uniqueness results for explosive solutions. Since it can have its own interest, we present it here.

First of all, we recall that any  $C^2$  function  $v_0$  solution of (2.26) satisfies (2.28) and (2.29). Moreover, by Lemma 2.2 we have that  $\omega_0 = e^{-\beta v_0} \in C^1(\bar{S})$  and  $\partial_\nu \omega_0 < 0$  on  $\partial S$ , hence, using that  $\nabla v_0 = -\frac{e^{\beta v_0}}{\beta} \nabla \omega_0$  and the estimate (2.28) we conclude that there exists a constant  $\sigma > 0$  such that, in a neighborhood of  $\partial S$

$$|\nabla v_0| \geq \frac{\sigma}{\rho(x)}. \quad (2.31)$$

In addition, it is possible to deduce from (2.28)–(2.29) that there exists a constant  $C_0 > 0$  such that

$$|D^2 v_0| \leq \frac{C_0}{\tilde{\rho}^2(x)} \quad \forall x \in S. \quad (2.32)$$

Indeed, take  $x_0 \in S$  and let  $\rho_0 = \frac{\rho(x_0)}{2}$ , where we recall that  $\rho(x_0) = \text{dist}(x_0, \partial S)$ . Then consider (in a local neighborhood of  $x_0$ ) the rescaled function

$$u_0(\xi) = v_0(x_0 + \rho_0 \xi) + \frac{\ln \rho_0}{\beta},$$

for  $\xi \in B(0, 1)$ . Note that  $\rho(x_0 + \rho_0 \xi) \in (\rho_0, 3\rho_0)$  so that (2.29) and (2.31) imply  $\frac{\sigma}{3} \leq |Du_0| \leq L_0 + L_1 \rho_0$ . Since  $v_0$  is a solution of (2.26), a simple scaling in the local coordinates gives that  $u_0$  is a solution of

$$-\Delta_g u_0 - (p-2) \frac{D^2 u_0 \nabla u_0 \cdot \nabla u_0}{\rho_0^2 + |\nabla u_0|^2} + \beta(p-1) |\nabla u_0|^2 = -\lambda_0 \rho_0^2 \quad \text{for } \xi \in B(0, 1)$$

with a slight abuse of notation since now, in local coordinates, the derivatives are taken with respect to the variable  $\xi$ . Since the second order operator is uniformly elliptic (as in (2.13)–(2.14)), by the classical regularity theory (e.g. see [14], Theorem 13.6 to deduce the Hölder estimates for  $Du_0$  and then apply the Schauder estimates, Chapter 6) we have that

$$|D^2 u_0(\xi)| \leq C \quad \forall \xi \in B\left(0, \frac{1}{2}\right)$$

where  $C$  is a constant depending on  $\sup_{B(0,1)} (|u_0| + |Du_0|)$ . Using the estimates (2.28)–(2.29) we can bound this last quantity only depending on  $M^*$ ,  $L_0$ ,  $L_1$ , hence we conclude that  $|D^2 u_0(0)| \leq C$ , which gives (2.32).

Now, take two solutions  $v_1, v_2$  of (2.26) corresponding to  $\lambda_1, \lambda_2$  with, say,  $\lambda_1 \leq \lambda_2$ . We adapt now an argument in [17]: consider the function  $\hat{v} = \theta v_2$ , for  $\theta < 1$ , and compute

$$\begin{cases} -\Delta_g \hat{v} - (p-2) \frac{D^2 \hat{v} \nabla \hat{v} \cdot \nabla \hat{v}}{1 + |\nabla \hat{v}|^2} + \beta(p-1) |\nabla \hat{v}|^2 = -\theta \lambda_2 \\ \quad + (1-\theta^2) \theta (p-2) \frac{D^2 v_2 \nabla v_2 \cdot \nabla v_2}{(1+|\nabla v_2|^2)(1+\theta^2|\nabla v_2|^2)} - (1-\theta) \theta \beta (p-1) |\nabla v_2|^2 \end{cases}$$

Using (2.29), (2.32) and (2.31), we know that

$$\left| \frac{D^2 v_2 \nabla v_2 \cdot \nabla v_2}{(1+|\nabla v_2|^2)(1+\theta^2|\nabla v_2|^2)} \right| \leq C \quad \text{in } S$$

hence  $\hat{v}$  satisfies

$$-\Delta_g \hat{v} - (p-2) \frac{D^2 \hat{v} \nabla \hat{v} \cdot \nabla \hat{v}}{1 + |\nabla \hat{v}|^2} + \beta(p-1) |\nabla \hat{v}|^2 \leq -\lambda_1 - (1-\theta) [\beta(p-1) \theta |\nabla v_2|^2 - C(\theta + \theta^2)(p-2) - \lambda_1].$$

Thanks to (2.31), we deduce that there exists  $\delta > 0$ , independent on  $\theta$ , such that  $\hat{v}$  satisfies

$$-\Delta_g \hat{v} - (p-2) \frac{D^2 \hat{v} \nabla \hat{v} \cdot \nabla \hat{v}}{1 + |\nabla \hat{v}|^2} + \beta(p-1) |\nabla \hat{v}|^2 \leq -\lambda_1$$

in  $\{x \in S : \rho(x) < \delta\}$ . However, from the estimate (2.28) which holds for  $v_1$  and  $v_2$  we have that  $v_1 - \hat{v} \rightarrow +\infty$  as  $\rho(x) \rightarrow 0$ , hence  $v_1 - \hat{v}$  has a minimum in  $\{x \in S : \rho(x) < \delta\}$

and, by standard maximum principle, it is reached when  $\rho(x) = \delta$ . Letting  $\theta \rightarrow 1$ , we conclude that

$$\min\{(v_1 - v_2)(x), x : \rho(x) \leq \delta\} = \min\{(v_1 - v_2)(x), x : \rho(x) = \delta\}.$$

On the other hand, looking at the equations of  $v_1, v_2$  in  $\{x \in S : \rho(x) > \delta\}$ , we also know (again by maximum principle) that

$$\min\{(v_1 - v_2)(x), x : \rho(x) \geq \delta\} = \min\{(v_1 - v_2)(x), x : \rho(x) = \delta\}$$

hence  $v_1 - v_2$  should have a global minimum reached at a point  $x_0 \in S$  such that  $\rho(x_0) = \delta$ . Since  $x_0$  lies inside the domain, and the function  $z = v_1 - v_2$  satisfies a smooth elliptic equation around  $x_0$ , using the strong maximum principle we conclude that  $v_1 - v_2$  is constant. This proves the uniqueness, up to a constant, of the solution of (2.26), and at the same time also the uniqueness of the ergodic constant ( $\lambda_1 = \lambda_2$ , as we already proved before).  $\square$

**Remark 2.3** The argument used in the last step of the previous proof also provides a general uniqueness result for explosive solutions of

$$\begin{cases} -\Delta_g v - (p-2) \frac{D^2 v \nabla v \cdot \nabla v}{1 + |\nabla v|^2} + \beta(p-1) |\nabla v|^2 + \epsilon v = f & \text{in } S \\ \lim_{x \rightarrow \partial S} v(x) = \infty. \end{cases} \quad (2.33)$$

Precisely, if  $f$  is a Lipschitz function, and  $\epsilon > 0$ , the problem (2.33) has a unique solution  $v \in C^2(S)$ . To our knowledge, such a result is new even in the euclidean setting  $M = \mathbb{R}^N$ .

We proceed now studying how the ergodic constant  $\lambda_\beta$  depends on  $\beta$ , which will lead to the proof of Theorem B.

**Proposition 2.4** *Under the assumptions of Theorem 2.1, the mapping  $\beta \mapsto \lambda_\beta$  is continuous and decreasing from  $(0, \infty)$  in  $(0, \infty)$ , and it verifies*

$$\lim_{\beta \rightarrow 0} \lambda_\beta = \infty. \quad (2.34)$$

*Proof. Step 1: the monotonicity.* Let  $0 < \beta_1 < \beta_2$  and let  $v_{\epsilon,1}$  and  $v_{\epsilon,2}$  be the corresponding solutions of (2.2) with  $\beta$  respectively replaced by  $\beta_1$  and  $\beta_2$ . Since the  $v_{\epsilon,i}$  are limit of solutions with finite boundary value there holds  $v_{\epsilon,1} > v_{\epsilon,2}$  by comparison principle. Therefore

$$\lambda_{\beta_1} := \lim_{\epsilon \rightarrow 0} \epsilon v_{\epsilon,1} \geq \lambda_{\beta_2} := \lim_{\epsilon \rightarrow 0} \epsilon v_{\epsilon,2}.$$

Next, if we assume that there exist  $\beta_i$  ( $i = 1, 2$ ) such that  $0 < \beta_1 < \beta_2$  and  $\lambda_{\beta_1} = \lambda_{\beta_2} = \lambda$  and if  $\omega_1$  and  $\omega_2$  are the corresponding solutions of (2.27) with  $\beta = \beta_i$  and  $\lambda = \lambda_{\beta_1} = \lambda_{\beta_2}$ , then (2.28) implies

$$m^{-1} \rho(x) \leq \omega_i \leq m \rho(x) \quad \forall x \in S,$$

for some  $m > 0$ . Set  $\tilde{\omega} = \omega_1^{\beta_2/\beta_1}$ , then

$$\begin{aligned} & -\operatorname{div}_g \left( (\beta_2^2 \tilde{\omega}^2 + |\nabla \tilde{\omega}|^2)^{p/2-1} \nabla \tilde{\omega} \right) - \beta_2 \lambda (\beta_2^2 \tilde{\omega}^2 + |\nabla \tilde{\omega}|^2)^{p/2-1} \tilde{\omega} \\ & = (p-1) \left( 1 - \frac{\beta_2}{\beta_1} \right) \left( \frac{\beta_2}{\beta_1} \right)^{p-1} \omega_1^{(p-1)(\beta_2/\beta_1-1)} (\beta_1^2 \omega_1^2 + |\nabla \omega_1|^2)^{(p-2)/2} \frac{|\nabla \omega_1|^2}{\omega_1}. \end{aligned} \quad (2.35)$$

Therefore  $\tilde{\omega}$  is a strict sub-solution. By homogeneity, and since  $\partial_\nu \tilde{\omega}$  vanishes on  $\partial S$ , we can assume that  $\tilde{\omega} \leq \omega_2$ , that there exists  $x_0 \in S$  such that  $\tilde{\omega}(x_0) = \omega_2(x_0)$  and the coincidence set of  $\tilde{\omega}$  and  $\omega_2$  is a subset of  $S$ . Let

$$z = -\frac{1}{\beta_2} (\ln \omega_2 - \ln \tilde{\omega}) = v_2 - \tilde{v}.$$

Then  $z \leq 0$ , it is not identically zero,  $z(x_0) = 0$  and  $z(x) \rightarrow -\infty$  as  $\rho(x) \rightarrow \partial S$ . Because

$$\begin{aligned} & -\Delta_g v_2 - (p-2) \frac{D^2 v_2 \nabla v_2 \cdot \nabla v_2}{1 + |\nabla v_2|^2} + \beta_2 (p-1) |\nabla v_2|^2 \\ & \leq -\Delta_g \tilde{v} - (p-2) \frac{D^2 \tilde{v} \nabla \tilde{v} \cdot \nabla \tilde{v}}{1 + |\nabla \tilde{v}|^2} + \beta_2 (p-1) |\nabla \tilde{v}|^2 \end{aligned}$$

developing this inequality, we obtain that, at  $x = x_0$ , there holds

$$\begin{aligned} & -\Delta_g z - (p-2) \frac{D^2 z \nabla v_2 \cdot \nabla v_2}{1 + |\nabla v_2|^2} \\ & + (p-2) \left[ \frac{D^2 \tilde{v} \nabla \tilde{v} \cdot \nabla \tilde{v}}{1 + |\nabla \tilde{v}|^2} - \frac{D^2 \tilde{v} \nabla v_2 \cdot \nabla v_2}{1 + |\nabla v_2|^2} \right] + \beta_2 (p-1) [|\nabla v_2|^2 - |\nabla \tilde{v}|^2] \leq 0 \end{aligned}$$

Since  $\tilde{v}, v_2$  are  $C^2$  in  $S$ , the strong maximum principle yields a contradiction. Therefore  $\beta \mapsto \lambda_\beta$  is decreasing.

*Step 2: the continuity.* Let  $\{\beta_n\}$  be a positive sequence such that  $\beta_n \rightarrow \beta_0$  and  $v_{\beta_n}$  be the corresponding solution of

$$\begin{cases} -\Delta_g v_{\beta_n} - (p-2) \frac{D^2 v_{\beta_n} \nabla v_{\beta_n} \cdot \nabla v_{\beta_n}}{1 + |\nabla v_{\beta_n}|^2} + \beta_n (p-1) |\nabla v_{\beta_n}|^2 = -\lambda_{\beta_n} & \text{in } S \\ \lim_{x \rightarrow \partial S} v_{\beta_n}(x) = \infty, \end{cases} \quad (2.36)$$

and let  $v_{\epsilon, \beta_n}$  be the corresponding solutions of (2.2) with  $\beta = \beta_n$ . Since  $\epsilon v_{\epsilon, \beta_n}$  remains locally bounded in  $S$  when  $\beta_n$  remains in a compact subset of  $(0, \infty)$  and converges to  $\lambda_{\beta_n}$  locally uniformly as  $\epsilon \rightarrow 0$ , the set  $\{\lambda_{\beta_n}\}$  is bounded. Up to a subsequence (not relabeled) we can assume that  $\lambda_{\beta_n} \rightarrow \bar{\lambda}$  as  $n \rightarrow \infty$ . Thanks to (2.28) and (2.29), there holds

$$\left| v_{\beta_n} + \frac{\ln \rho(x)}{\beta_n} \right| \leq C_0 \quad \text{and} \quad |\nabla v_{\beta_n}| \leq \frac{C_1}{\rho(x)}, \quad (2.37)$$

for some constants  $C_0, C_1$ , hence the sequence  $\{v_{\beta_n}\}$  remains locally bounded in  $W_{loc}^{1,\infty}(S)$  and, therefore, in  $C_{loc}^{2,\alpha}(S)$ . Up to a subsequence  $v_{\beta_n} \rightarrow \bar{v}$  in  $C_{loc}^2(S)$ , and  $\bar{v}$  is a solution of

$$\begin{cases} -\Delta_g \bar{v} - (p-2) \frac{D^2 \bar{v} \nabla \bar{v} \cdot \nabla \bar{v}}{1 + |\nabla \bar{v}|^2} + \beta_0 (p-1) |\nabla \bar{v}|^2 = -\bar{\lambda} & \text{in } S \\ \lim_{x \rightarrow \partial S} \bar{v}(x) = \infty. \end{cases}$$

By uniqueness of the ergodic limit,  $\bar{\lambda} = \lambda_{\beta_0}$ , and  $\lambda_{\beta_n} \rightarrow \lambda_{\beta_0}$  for the whole sequence.

*Step 3: (2.34) holds.* Let  $\omega$  be a positive solution of

$$\begin{cases} -\operatorname{div}_g ((\beta^2 \omega^2 + |\nabla \omega|^2)^{p/2-1} \nabla \omega) = \beta \lambda_\beta (\beta^2 \omega^2 + |\nabla \omega|^2)^{p/2-1} \omega & \text{in } S \\ \omega = 0 & \text{on } \partial S. \end{cases} \quad (2.38)$$

We normalize  $\omega$  by

$$\int_S |\nabla \omega|^p dv_g = 1.$$

Therefore, if  $\mu_S$  is the first eigenvalue of  $-\operatorname{div}_g (|\nabla \cdot|^{p-2} \nabla \cdot)$  in  $W_0^{1,p}(S)$ , there holds

$$\int |\omega|^p dv_g \leq \frac{1}{\mu_S}.$$

Multiplying (2.38) by  $\omega$  and integrating over  $S$  yields to

$$\int_S (\beta^2 \omega^2 + |\nabla \omega|^2)^{p/2} dv_g = \beta (\lambda_\beta + \beta) \int_S (\beta^2 \omega^2 + |\nabla \omega|^2)^{p/2-1} \omega^2 dv_g. \quad (2.39)$$

Clearly

$$\int_S (\beta^2 \omega^2 + |\nabla \omega|^2)^{p/2} dv_g \geq \int_S |\nabla \omega|^p dv_g = 1.$$

If  $p \geq 2$ ,

$$\begin{aligned} \int_S (\beta^2 \omega^2 + |\nabla \omega|^2)^{p/2-1} \omega^2 dv_g &\leq 2^{p/2-2} \int_S (\omega^p + \omega^2 |\nabla \omega|^{p-2}) dv_g \\ &\leq 2^{p/2-2} \left(1 + \frac{2}{p}\right) \int_S \omega^p dv_g + 2^{p/2-2} \left(1 - \frac{2}{p}\right) \int_S |\nabla \omega|^p dv_g \\ &\leq C_{p,S} \end{aligned}$$

This implies

$$\beta (\lambda_\beta + \beta) \geq \frac{1}{C_{p,S}} \implies \lambda_\beta \geq \frac{1}{C_{p,S} \beta} - \beta. \quad (2.40)$$

If  $1 < p < 2$ ,

$$\int_S \frac{\omega^2 dv_g}{(\beta^2 \omega^2 + |\nabla \omega|^2)^{1-p/2}} \leq \beta^{p-2} \int_S |\omega|^p dv_g \leq \frac{\beta^{p-2}}{\mu_S}.$$

Therefore

$$\beta^{p-1} (\lambda_\beta + \beta) \geq \mu_S \implies \lambda_\beta \geq \frac{\mu_S}{\beta^{p-1}} - \beta. \quad (2.41)$$

Clearly (2.40) and (2.41) imply (2.34).  $\square$

*Remark.* Using the uniform ellipticity and the maximum principle, (2.40) and (2.41) can possibly be improved in  $\lambda_\beta \geq \frac{C}{\beta}$ .

We have now all the ingredients for the proof of Theorem B.

**Proof of Theorem B.** If we set  $\omega = e^{-\beta v}$  where  $v$  is the solution of (2.1), then  $\omega$  is defined up to a multiplicative constant and satisfies (2.38). By Lemma 2.2,  $\omega \in C^1(\bar{S}) \cap C^2(S)$ . Therefore the Theorem is obtained if we can prove that there exists a unique  $\beta := \beta_s > 0$  such that

$$\lambda_\beta = \beta(p-1) + p - d - 1. \quad (2.42)$$

But the mapping  $\beta \mapsto \lambda_\beta - \beta(p-1)$  is continuous and decreasing on  $(0, \infty)$ . Clearly

$$\lim_{\beta \rightarrow \infty} \lambda_\beta - \beta(p-1) = -\infty,$$

and

$$\lim_{\beta \rightarrow 0} \lambda_\beta - \beta(p-1) = \infty,$$

by Proposition 2.4. The results follows by continuity.  $\square$

### 3 The regular case and Tolksdorf's result

If  $\beta < 0$ , the equation satisfied by a separable  $p$ -harmonic function  $u$  under the form (1.1) is unchanged. However, if we set  $\tilde{\beta} = -\beta$ , then (1.4) turns into

$$-div \left( (\tilde{\beta}^2 \omega + |\nabla' \omega|^2)^{p/2-1} \nabla' \omega \right) = \tilde{\beta}(\tilde{\beta}(p-1) + N - p)(\tilde{\beta}^2 \omega + |\nabla' \omega|^2)^{p/2-1} \omega. \quad (3.1)$$

Furthermore, if a solution  $\omega$  of (3.1) in  $S \subset S^{N-1}$  exists which vanishes on  $\partial S$ , then  $\tilde{\beta}(p-1) + N - p > 0$  by multiplying by  $\omega$  and integration over  $S$ . By setting

$$v = -\frac{\ln \omega}{\tilde{\beta}},$$

then  $v$  satisfies

$$\begin{cases} -div \left( (1 + |\nabla' v|^2)^{p/2-1} \nabla' v \right) + \beta(p-1) (1 + |\nabla' v|^2)^{p/2-1} |\nabla' v|^2 \\ \quad = -(\tilde{\beta}(p-1) + N - p) (1 + |\nabla' v|^2)^{p/2-1} & \text{in } S \\ \lim_{\sigma \rightarrow \partial S} v(\sigma) = \infty. \end{cases} \quad (3.2)$$

In the general setting of a Riemannian manifold, Theorem 2.1 and Proposition 2.4 are valid with  $\beta$  replaced by  $\tilde{\beta}$ . The proof of Theorem B holds except that (2.42) is replaced by

$$\lambda_{\tilde{\beta}} = \tilde{\beta}(p-1) + d + 1 - p. \quad (3.3)$$

Because the function  $\tilde{\beta} \mapsto \lambda_{\tilde{\beta}} - \tilde{\beta}(p-1)$  is unchanged, the proof of Theorem B applies and shows that there exists a unique  $\tilde{\beta} := \tilde{\beta}_s > 0$  such that (3.3) holds. Consequently we have proved the following result which contains Tolksdorf's initial result if  $(M, g) = (S^{N-1}, g_0)$ .



**Corollary 3.1** *Under the assumptions of Theorem 2.1 there exists a unique  $\tilde{\beta} := \tilde{\beta}_S > 0$  such that the problem*

$$\begin{cases} -\operatorname{div}_g \left( (\tilde{\beta}^2 \omega^2 + |\nabla \omega|^2)^{p/2-1} \nabla \omega \right) = \tilde{\beta} \left( \tilde{\beta}(p-1) + d + 1 - p \right) (\tilde{\beta}^2 \omega^2 + |\nabla \omega|^2)^{p/2-1} \omega & \text{in } S \\ \omega = 0 & \text{on } \partial S, \end{cases} \quad (3.4)$$

*admits a positive solution  $\omega \in C^1(\bar{S}) \cap C^2(S)$ . Furthermore  $\omega$  is unique up to an homothethy.*

## A Appendix

We prove here the  $C^{1,\gamma}$  regularity up to the boundary, stated in Lemma 2.2, for solutions of degenerate equations in divergence form

$$\begin{cases} -\operatorname{div} (a(x, u, \nabla u)) = B(x, u, \nabla u) & \text{in } S \\ u = 0 & \text{on } \partial S. \end{cases} \quad (A.1)$$

We will assume that  $a(x, s, \xi)$  satisfies the following conditions: there exist constants  $\lambda, \Lambda, \beta > 0$ , and  $\alpha \in (0, 1]$ , and a continuous function  $\mu : S \times \mathbb{R} \rightarrow \mathbb{R}$  such that, for every  $s, t \in \mathbb{R}$ , for every  $\xi, \eta \in \mathbb{R}^N$ , and a.e.  $x \in \Omega$ :

$$\frac{\partial a^i}{\partial \xi_j}(x, s, \xi) \eta_i \eta_j \geq \lambda (\mu(x, s)^2 + |\xi|^2)^{\frac{p-2}{2}} |\eta|^2, \quad (A.2)$$

$$\left| \frac{\partial a^i}{\partial \xi_j}(x, s, \xi) \right| \leq \Lambda (\mu(x, s)^2 + |\xi|^2)^{\frac{p-2}{2}}, \quad (A.3)$$

$$|a(x, s, \xi) - a(y, t, \xi)| \leq \beta (1 + |\xi|^{p-2} + |\xi|^{p-1}) [|x - y|^\alpha + |s - t|^\alpha], \quad (A.4)$$

$$|B(x, s, \xi)| \leq \beta (1 + |\xi|^p). \quad (A.5)$$

The model we have in mind is clearly

$$a(x, u, \nabla u) = (\mu(x, u)^2 + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u$$

where  $p > 1$ , and the function  $\mu(x, s)$  is Lipschitz (or possibly Hölder) continuous. In many cases, as in the proof of Lemma 2.2, the a priori information that  $u$  is Lipschitz (or Hölder) continuous could allow us to consider only the case  $\mu = \mu(x)$ .

The  $C^{1,\gamma}$  estimates, or similar kind of regularity results, are by now classical since the works of E. DiBenedetto [9] and P. Tolksdorf [20] for the  $p$ -Laplace equation: as far as the global regularity, up to the boundary, is concerned, we refer to the works of G. Lieberman (e.g. [18]) or to [8]. Despite a large amount of literature available, it seems that no exact reference applies to our model, so that, for the sake of completeness, we feel like giving a proof of this result, at least detailing the possible slight modifications in order that previous results can be generalized. To this purpose, we observe that while the case  $p \geq 2$  is somehow contained, if not in previous statements, at least in previous arguments

(specifically, we refer to [18]), this seems not sure for the case  $p < 2$  because of our growth assumption (A.4) (roughly speaking, the  $(x, s)$ -derivatives may grow like  $|\xi|^{p-2}$ ). Finally, we note that the next result would still hold for a nonhomogeneous boundary condition ( $u = \varphi$  on  $\partial S$ ) provided  $\varphi$  belongs to  $C^{1,\alpha}(\partial S)$ .

**Theorem A.1** *Let  $S$  be a bounded  $C^{1,\alpha}$  domain in  $\mathbb{R}^N$ , and assume that (A.2)–(A.5) hold true. If  $u$  is a bounded weak solution of (A.1), then there exists  $\gamma \in (0, 1)$  such that  $u \in C^{1,\gamma}(\bar{S})$  and moreover*

$$\|u\|_{C^{1,\gamma}(\bar{S})} \leq C(\Lambda/\lambda, \alpha, \|u\|_\infty, p, N, S).$$

**Proof.** Because our specific interest is in the boundary estimate, we only prove the regularity of  $u$  around a point  $x_0 \in \partial S$  (the inner regularity is treated in the same manner). Up to straightening the boundary, we can assume that locally  $\partial S = \{x : x_N = 0\}$  and  $S = \{x : x_N > 0\}$ .

We follow the standard approach via perturbation argument. We denote  $B_R = \{x : |x - x_0| < R\}$ ,  $B_R^+ = B_R \cap S$ , and consider the solution  $v$  of

$$\begin{cases} -\operatorname{div}(a(x_0, u(x_0), \nabla v)) = 0 & \text{in } B_R^+ \\ v = u & \text{on } \partial B_R^+. \end{cases} \quad (\text{A.6})$$

Problem (A.6) has a unique solution  $v \in W^{1,p}(B_R^+)$ . Due to assumptions (A.2)–(A.3), the estimates concerning  $v$  are well-established ([9], [20], [18]). In particular, from Lemma 5 in [18] we have, for some  $\sigma > 0$ ,

$$\operatorname{osc}_{B_r^+} \nabla v \leq C \left( \frac{r}{R} \right)^\sigma \left( R^{-N} \int_{B_R^+} |\nabla v|^p dx \right)^{\frac{1}{p}} \quad \forall r < \frac{R}{2} \quad (\text{A.7})$$

where  $C$ , here and after, depends only on the constants appearing in the hypotheses and possibly on  $\|u\|_\infty$ , in particular through the quantity  $\sup\{|\mu(x, s)|, x \in \bar{S}, |s| \leq \|u\|_\infty\}$ . Moreover, since  $a(x, s, \xi)\xi \geq c(|\xi|^p - |\mu|^p)$ , one easily deduces from (A.6), using  $v - u$  as test function and Young's inequality, that

$$\int_{B_R^+} |\nabla v|^p dx \leq C \left( 1 + \int_{B_R^+} |\nabla u|^p dx \right). \quad (\text{A.8})$$

Finally, the maximum principle gives  $\inf_{\partial B_R^+} u \leq v \leq \sup_{\partial B_R^+} u$ , which yields

$$\operatorname{osc}_{B_R^+} v \leq \operatorname{osc}_{B_R^+} u. \quad (\text{A.9})$$

Now take  $u - v$  as test function both in (A.1) (restricted to  $B_R^+$ ) and in (A.6) to obtain

$$\int_{B_R^+} a(x, u, \nabla u) \nabla(u - v) dx - \int_{B_R^+} a(x_0, u(x_0), \nabla v) \nabla(u - v) dx = \int_{B_R^+} B(x, u, \nabla u)(u - v) dx.$$

Denote  $D_v := \{x \in B_R^+ : |\nabla u| < |\nabla v|\}$  and  $D_u := \{x \in B_R^+ : |\nabla v| \leq |\nabla u|\}$ : hence we have

$$\begin{aligned}
& \int_{D_v} [a(x, u, \nabla u) - a(x, u, \nabla v)] \nabla(u - v) dx \\
& + \int_{D_u} [a(x_0, u(x_0), \nabla u) - a(x_0, u(x_0), \nabla v)] \nabla(u - v) dx \\
& = \int_{D_v} [a(x_0, u(x_0), \nabla v) - a(x, u, \nabla v)] \nabla(u - v) dx \\
& + \int_{D_u} [a(x_0, u(x_0), \nabla u) - a(x, u, \nabla u)] \nabla(u - v) dx + \int_{B_R^+} B(x, u, \nabla u)(u - v) dx
\end{aligned} \tag{A.10}$$

Using (A.4) and the definition of  $D_v$ , we have

$$\begin{aligned}
& \int_{D_v} [a(x_0, u(x_0), \nabla v) - a(x, u, \nabla v)] \nabla(u - v) dx \\
& \leq 2\beta \int_{D_v} (1 + |\nabla v|^{p-2} + |\nabla v|^{p-1}) |\nabla v| [|x - x_0|^\alpha + |u(x) - u(x_0)|^\alpha] dx \\
& \leq C[R^\alpha + (\text{osc}_{B_R^+} u)^\alpha] \int_{D_v} (1 + |\nabla v|^p) dx
\end{aligned}$$

Similarly we estimate the second term in the right hand side of (A.10), and using also (A.5) we deduce

$$\begin{aligned}
& \int_{D_v} [a(x, u, \nabla u) - a(x, u, \nabla v)] \nabla(u - v) dx \\
& + \int_{D_u} [a(x_0, u(x_0), \nabla u) - a(x_0, u(x_0), \nabla v)] \nabla(u - v) dx \\
& \leq C[R^\alpha + (\text{osc}_{B_R^+} u)^\alpha + \text{osc}_{B_R^+} u] \int_{B_R^+} (1 + |\nabla v|^p + |\nabla u|^p) dx,
\end{aligned}$$

where we used that  $\text{osc}_{B_R^+}(u - v) \leq 2 \text{osc}_{B_R^+} u$  thanks to (A.9).

Now, in both terms in the left hand side we use (A.2) which implies, for every  $(x, s, \xi)$ :

$$[a(x, s, \xi) - a(x, s, \eta)] \cdot (\xi - \eta) \geq c(\lambda) (\mu(x, s)^2 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} |\xi - \eta|^2. \tag{A.11}$$

If  $p < 2$  we get (recall that the generic constant  $C$  may depend on  $\|u\|_\infty$ )

$$\begin{aligned}
& \int_{D_v} [a(x, u, \nabla u) - a(x, u, \nabla v)] \nabla(u - v) dx \\
& + \int_{D_u} [a(x_0, u(x_0), \nabla u) - a(x_0, u(x_0), \nabla v)] \nabla(u - v) dx \\
& \geq C \int_{D_v \cup D_u} [1 + |\nabla u|^2 + |\nabla v|^2]^{\frac{p-2}{2}} |\nabla(u - v)|^2 dx
\end{aligned}$$

hence using Hölder inequality we end up with

$$\int_{B_R^+} |\nabla(u - v)|^p dx \leq C[R^\alpha + (\text{osc}_{B_R^+} u)^\alpha + \text{osc}_{B_R^+} u]^q \int_{B_R^+} (1 + |\nabla v|^p + |\nabla u|^p) dx,$$

with  $q = \frac{p}{2}$ . If  $p \geq 2$  we simply get rid of the term  $\mu^2$  in (A.11) and obtain the same inequality with  $q = 1$ . Therefore, using also (A.8), we conclude for any  $p > 1$

$$\int_{B_R^+} |\nabla(u - v)|^p dx \leq C[R^\alpha + (\text{osc}_{B_R^+} u)^\alpha + \text{osc}_{B_R^+} u]^q \int_{B_R^+} (1 + |\nabla u|^p) dx \tag{A.12}$$

with  $q = \min(1, \frac{p}{2})$ .

Starting from the inequality (A.12) it is possible to deduce the Hölder regularity of  $\nabla u$  following well-known arguments. In particular, if  $u$  is Lipschitz continuous (as in our application in Lemma 2.2) the conclusion is straightforward, since (A.12) implies

$$\int_{B_R^+} |\nabla(u - v)|^p dx \leq CR^{N+\alpha q}$$

and (A.7)–(A.8) give that  $\operatorname{osc}_{B_r^+} \nabla v \leq C (r/R)^\sigma$ .

Then, defining  $(F)_r = \frac{1}{|B_r^+|} \int_{B_r^+} F(y) dy$  for  $F = \nabla u$  or  $\nabla v$ , we deduce

$$\begin{aligned} \int_{B_r^+} |\nabla u - (\nabla u)_r|^p dx &\leq C \left[ \int_{B_r^+} |\nabla u - \nabla v|^p + \int_{B_r^+} |\nabla v - (\nabla v)_r|^p \right] \\ &\leq C [R^{N+\alpha q} + r^N \left(\frac{r}{R}\right)^{\sigma p}] \end{aligned}$$

and choosing  $R = r^\theta$  for some suitable  $\theta < 1$  the conclusion follows from the results of Campanato [7].

In the general case, i.e. when a Lipschitz estimate on  $u$  is not available, one need further work to estimate the right hand side of (A.12). For this purpose, starting from (A.12), we can follow the arguments of G. Lieberman ([18], Section 3) and still get at the conclusion. ■

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